Al-Kitab Journal for Pure Sciences
ISSN: 2617-1260 (print), 2617-8141(online)
https://isnra.net/index.php/kjps

Special Methods Controllability and Observability in Optimal Control Systems

Ali Farhan Hashoosh*

*Department of Mathematics, College of Basic Education, Misan University, Iraq

Corresponding Author: ali_fr@uomisan.edu.iq

ORCID: 0000-0002-3910-6852

Citation: Hashoosh AF. Special Methods Controllability and Observability in Optimal Control Systems. Al-Kitab J. Pure Sci. [Internet]. 2024 July. 06 [cited 2024 July. 06];8(2):78-93. Available from: https://doi.org/10.32441/kjps.08.02.p7.

Keywords: Mathematical Modeling, Optimal Control, Controllability, Observability.

Article History

Received	15 Mar.	2024
Accepted	25 Jun.	2024
Available online	06 July	2024

Abstract:

This article explores a specific case study that examines controllability and observability in general and in specific so that it is easy for the reader to understand these two concepts, which are fundamental in optimal control theory. Algorithms have been written to determine the controllability and observability of optimal control systems using the MATLAB programming language, and new techniques have been developed to deal with them. In addition, a critical test was created in which the state variables of the system or, more precisely, their corresponding states were split together, illustrating this with more than one example. Divided into four groups in a linear manner, as the article explains. To understand the controllability and observability of some more complex systems, this article is a starting point for the future expansion of these two concepts through the development of new algorithms or other applied solution methods or the creation of new algorithms.

Keywords: Mathematical Modeling, Optimal Control, Controllability, Observability.

طرق خاصة لقابلية التحكم وقابلية الملاحظة في أنظمة التحكم الأمثل

علي فرحان حاشوش*

* قسم الرياضيات، كلية التربية الأساسية ،جامعة ميسان، العراق
ali_fr@uomisan.edu.iq

الخلاصة:

تستكشف هذه المقالة دراسة حالة محددة تدرس إمكانية التحكم وإمكانية الملاحظة على نطاق واسع ودقيق بحيث يسهل على القارئ فهم هذين المفهومين، وهما مفهومان أساسيان في نظرية التحكم الأمثل. تمت كتابة خوارزميات لتحديد إمكانية التحكم وقابلية الملاحظة لأنظمة التحكم المثلى باستخدام لغة البرمجة *MATLAB*، كما تم تطوير تقنيات جديدة للتعامل معها. بالإضافة إلى ذلك، تم إنشاء اختبار نقدي تم فيه تقسيم متغيرات حالة النظام، أو بشكل أكثر دقة، الحالات المقابلة لها، مع توضيح ذلك بأكثر من مثال. مقسمة إلى أربع مجموعات بشكل خطي، كما يوضح المقال. من أجل فهم إمكانية التحكم وإمكانية الملاحظة في بعض الأنظمة الأكثر تعقيدًا، تعد هذه المقالة نقطة انطلاق للتوسع المستقبلي لهذين المفهومين من خلال تطوير خوارزميات جديدة أو طرق حل تطبيقية أخرى أو إنشاء خوارزميات جديدة.

الكلمات المفتاحية: نماذج رياضية، التحكم الامثل، قابلية التحكم، قابلية الملاحظة.

1. Introduction:

Controllability and observability are two basic and important concepts in modern control theory. The stability of the control system and its types is also stable (Routh stability criterion and Lyapunov) [1] [2]. In 1960, Kalman defined these two ideas in order [2] to determine the degree to which a system can be observed and managed [3]. The following fundamental queries must be addressed for any control system, especially for multivariable systems:

a) Is it possible to find a control function $u(t)$ that will, in a finite amount of time, change the system's initial state (x_t) into the desired final state (x_f)?

b) Is it possible to assess the system's state by analyzing its performance over a limited time?

The terms controllability and observability refer to the two ideas at play. Accordingly, the system is controllable if the first question has a "yes" response. The system is also observable if the second question has a yes response. It is important to acknowledge the fundamental nature of these problems. For instance, it makes no sense to attempt controlling a system by feedback of a state variable that permits the system's poles to be positioned arbitrarily unless the system is controllable. Similarly, attempting to reconstruct unmeasurable state variables of the system using so-called observers is futile unless the system is observable. In reality, controllability and observability are two dual concepts that are intimately associated with the cancellation of zeros and poles in the transfer function of the system [5].

2. Controllability

We say about a system that it is controllable if and only if it is possible through the control vector to bring the system from the initial state $x(t_0) = x_0$o any final state $x(t_f) = x_f$ within a specified time $t > 0$.

In the case of nonlinear systems, these equations take the following form:

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

$$y = C(t)x(t) + D(t)u(t)$$

In the case of linear systems fixed with time, the equations take the following form:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y = Cx(t) + Du(t)$$

Where

A is the state matrix of order $n \times n$, B is cthe ontrol matrix of order $n \times m$, C is the output matrix of order $1 \times n$, D is Direct transfer matrix of order $1 \times m$, n the number of state vector state.,m the number of control vector states [4]. We consider a system described by the state equations:

$$\dot{x} = Ax(t) + Bu(t) \qquad (2.1)$$

$y = Cx(t)$

With the transformation:

$$x(t) = pz(t) \qquad (2.2)$$

we can transform equation (2.1) into the form:

$$\dot{z} = A_1 z(t) + B_1 u(t) \qquad (2.3)$$

$$y = C_1 z(t)$$

Where $A_1 = P^{-1}AP$, $B_1 = P^{-1}B$ and $C_1 = CP$. Assuming that A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ we can choose P so that A_1 is a diagonal matrix, that is,

$$A_1 = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

If $n = m = 2$, the first of the equations (2.3) has the form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Which is written as

$\dot{z}_1 = \lambda_1 z_1 + b_1^T u$ (**2.4**)

$\dot{z}_2 = \lambda_2 z_2 + b_1^T u$

Where b_1^T and b_2^T are the row vectors of the matrix B_1

The output equation is

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$

Which can be written as

$y_1 = c_{11}z_1 + c_{12}z_2$

$y_2 = c_{21}z_1 + c_{22}z_2$

Or

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}c_{11}\\ c_{21}\end{bmatrix} z_1 + \begin{bmatrix}c_{12}\\ c_{22}\end{bmatrix} z_2$$

So that

$y = c_1 z_1 + c_2 z_2$ (**2.5**)

where c_1 and c_2 are the column vectors of C_1. So, in general, equation (**2.3**) can be written in the form:

$\dot{z}_i = \lambda_i z_i + b_i^T u(t)$ $(i = 1,2,3,\dots,n)$

$y = \sum_{i=1}^{n} c_i z_i$ (**2.6**)

It is seen from equation (**2.6**) that if b_i^T the i^{th} h row of B_1 has all zero components, then

$$\dot{z}_i = \lambda_i z_i + 0$$

And the input $u(t)$ has no influence on the i^{th} mode of the system. The mode is said to be uncontrollable, and a system having one or more such modes is uncontrollable **[6][7]**.

Example 1: Check whether the system having the state-space representation

$$\dot{x} = \begin{bmatrix}-1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}4\\ 6\end{bmatrix} u$$

$$y = \begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Is controllable?

Solution: The characteristic equation is

$|\lambda I - A| = \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\Longrightarrow \lambda = 1 \;\&\; \lambda = 2$

The corresponding eigenvectors are

$x_1 = [1 \quad 1]^T \; and \; x_2 = [2 \quad 3]^T$

so that the modal matrix is

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \; and \; P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [-1 \quad -4] z$$

This equation shows that the first mode is uncontrollable and so the system is uncontrollable. On the basis of the above result, we now derive an extremely useful criterion for determining whether a system is controllable. Although at this stage we consider only the necessity of this criterion, it is also a sufficient condition. To simplify the notation and the mathematical manipulations, we consider a SISO (single-input and single-output) system, so that in equation (2.1) B is a one-column matrix, that is a column vector b (say), and C is a row vector$\acute{c}_1$. The result holds for the more general case when the system is multivariable.

Equations (2.1) and (2.3) are then written as

$\dot{x} = Ax(t) + bu(t)$ (2.1.a)

$y = Cx(t)$

And

$\dot{z} = A_1 z(t) + b_1 u(t)$ (2.3.a)

$y = \acute{c}_1 z(t)$

We have chosen an indirect way of deriving the controllability criterion. It has the advantage of simplicity, but a penalty we paid or this is some loss in the logic behind the setting up of the criterion. We have established that the necessary condition for the system defined by equation (2.1. a) to be controllable is that the components of the vector

$b_1 = [\beta_1 \quad \beta_2 \quad \dots \quad \beta_n]^T$ in equation (2.3.a)are all non-zero

In equation (2.3. a) the matrix $A_1 = diag\{\lambda_1, \lambda_2. \dots, \lambda_n\}$where the eigenvalues

$\lambda_1, \lambda_2. \dots, \lambda_n$ are assumed distinct. Hence the matrix:

$$\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{bmatrix}$$

Has linearly independent columns, so that it is non-singular. It follows that the necessary condition to be controllable is that the (partitioned) matrix:

$$Q_p = [b_1 \quad A_1b_1 \quad A_1^2b_1 \quad \cdots \quad A_1^{n-1}b_1] = \begin{bmatrix} \beta_1 & \lambda_1\beta_1 & \dots & \lambda_1^{n-1}\beta_1 \\ \beta_2 & \lambda_2\beta_2 & \dots & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \lambda_n\beta_n & \dots & \lambda_n^{n-1}\beta_n \end{bmatrix} \qquad (\mathbf{2.7})$$

The matrix $Q_p$is non-singular.

Since

$$A_1 = P^{-1}AP \text{ and } b_1 = P^{-1}b$$

We have

$A_1b_1 = P^{-1}AP\,P^{-1}b = P^{-1}Ab$

$A_1^2b_1 = P^{-1}A^2P\,P^{-1}b = P^{-1}A^2b$

$\vdots$

$A_1^{n-1}b_1 = P^{-1}A^{n-1}P\,P^{-1}b = P^{-1}A^{n-1}b$

So that

$$Q_p = P^{-1}[b \quad Ab \quad A^2B \quad \cdots \quad A^{n-1}b] = P^{-1}Q_c$$

Where

$Q_c = [b \quad Ab \quad A^2B \quad \cdots \quad A^{n-1}b]$ $\qquad (\mathbf{2.8})$

Since Q_p(for a controllable system) and P^{-1} are both non-singular, Q_c (for a controllable system) is also non-singular.

As Q_p is non-singular, its n columns are linearly independent. So that the rank of the matrix Q_c written as $r(Q_c)$ is n.

2.1. Controllability Test

To find out the controllability of a system, consider a system described by the state equations:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y = Cx(t)$$

Step 1: write the matrix $Q_c(\,is\ called\ the\ system\ controllability\ matrix)$

$$Q_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

Step 2: find the determinant of Q_c if it is not equal to zero then the control system is controllable or if the determinant of Q_c equal to zero then the control system is uncontrollable.

If the matrix A in system order is higher than 3×3, it is difficult to know whether the system is controllable or not in a previous way, so we assume that

$$\Phi = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-m}B]$$

When m is number of inputs

Step 1: write the matrix Q_c($is\ called\ the\ system\ controllability\ matrix$)

$$\Phi = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-m}B]$$

Step 2: find rank of Φ if it is equal to n then the control system is controllable or if rank of Φ not equal to n then the control system is uncontrollable.

Or we can calculate the determinant $\Phi\Phi^T$ if it is not equal to zero then the control system is controllable or if determinant of $\Phi\Phi^T$ equal to zero then the control system is uncontrollable.

Example 2: Verify the controllability of control system which is the presented by state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

Solution:

Step 1: $\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & -4 \end{bmatrix}$

Step 2: $rank(\Phi) = 4 = \mathrm{n}$

Then the system is controllable.

2.2. Program for finding controllability

A program was developed in MATLAB to create controllability, and it was saved under the name “ctrb” and can be used when needed.

The program:

$Function\ co = ctrb(A, B)$

$N = length(A);$

$co = ctrb(a, b);$

$if\ rank(co) \cong n$

$disp('no\ contrable')$

$else$

$disp('\ contrable')$

end

Example 3: Is the system given as follows controllable?

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\x_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\-1&-2&0\\1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u$$

$$y(t)=\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Solution: Using the preceding method results in the following

$\gg A=[-1\quad 0\quad 0;-1\quad -2\quad 0;1\quad 0\quad 0];$

$\gg B=[1;0;0];$

$\gg co(A,B)$

$contable$

$ans =$

$\begin{matrix}1&-1&1\\0&-1&3\\0&1&-1\end{matrix}$

That is, the system is controllable and the value of the matrix Q_c:

$$Q_c=\begin{bmatrix}1&-1&1\\0&-1&3\\0&1&-1\end{bmatrix}$$

3. Observability

A system is said to be observable if the initial vector $x(t)$ can be found from the measurement of $u(t)$ and $y(t)$. The plant described by (**2.1**) is completely state observable if the inverse matrix exists **[8]**.

By using the transform $x = Pz(x)$ as in the (**2.1**) section, we end up with the system state equations in the form of equation (**6.3**), that is

$$\dot{z}=A_1z(t)+B_1u(t)$$

$y=C_1z(t)$

If a row of the matrix C_1 is zero, the corresponding mode of the system will not appear in the output y. In this case the system is unobservable, since we cannot determine the state variable corresponding to the row of zeros in C_1 from y.

Example 4: Check whether the system having the state-space representation

$$\dot{x} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is observable?

Solution: The characteristic equation is

$|\lambda I - A| = \lambda^2 - 1 = 0$

$(\lambda - 1)(\lambda + 1) = 0$

$\Longrightarrow \lambda = 1 \;\&\; \lambda = -1$

The corresponding eigenvectors are

$x_1 = [1 \quad 1]^T$ and $x_2 = [2 \quad 3]^T$

so that the modal matrix is

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Using the transformation $x = Pz$, the state-equation becomes

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Then the system is unobservable.

Example 5: illustrates the importance of the observability concept. In this case, we have an unstable system, whose instability is not observed in the output measurement. The dual controllability concept is of equal theoretical importance. An uncontrollable system has one or more modes that are not influenced by the input.

We now similarly derive a criterion for observability to that used to derive the controllability criterion.

Again, for simplicity, we consider a $SISO$ system, but the result holds for the more general multivariable system. It can be seen that the necessary conditions for systems defined by equation (**1.a**) to be observable is that the components of the vector$b_1 = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n]^T$ in equation (**3.a**) are all non-zero

for a controllable system we have the matrix

$$Q_1 = \begin{bmatrix} c_1^T \\ c_1^T A_1 \\ \vdots \\ c_1^T A_1^{n-1} \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_2 \lambda_1 & \gamma_2 \lambda_2 & \cdots & \gamma_n \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n \lambda_1^{n-1} & \gamma_2 \lambda_2^{n-1} & \cdots & \gamma_n \lambda_n^{n-1} \end{bmatrix} \qquad (3.1)$$

The matrix $Q_1$is non-singular.

Since

$$A_1 = P^{-1}AP \; and \; C_1^t = c^T P$$

We have

$$c_1^T A_1 = c^T P P^{-1} A P = c^T A P$$
$$A_1^2 b_1 = c^T P P^{-1} A^2 P = c^T A^2 P$$
$$\vdots$$
$$A_1^{n-1} b_1 = c^T P P^{-1} A^{n-1} P = c^T A^{n-1} P$$

So that

$$Q_1 = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} P = Q_o P$$

Where

$$Q_o = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} \qquad (3.2)$$

Since Q_o(for an observable system) and P are both non-singular, Q_o (for a observable system) is also non-singular.

3.1. The observability criterion

In equation (3.1) if the rank of matrix Q_o is n, the system can be called observable system. If rank Q_o less than n ,the system is un observable.

3.2. Observability test

A control system is said to be observable if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

To find out whether the control system is observable or not, we use a Kalman's test:

Step 1: form the matrix $Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T]$

Step 2: Take determinant of Q_o if it is not equal to zero then the control system is observable or if determinant of Q_c equal to zero then the control system is not observable [9].

Example 5: Verify the observability of control system which is the presented by state equation:

$$\dot{x} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution: Given $A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $and \; c = [1 \quad 1]$ $, n = 2$

Step 1: $Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T] = [C^T \quad A^T C^T] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$

Step 2: $|Q_o| = \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$

the system is observable.

3.3. Program for finding observability

MATLAB program was designed to find observability and was saved under the name "obsv" It is used when needed, as shown below [10]:

$function\ ob = \ obsv(A, C)$

$\%'The\ function\ ob = obsv(A, C) returns\ the\ transformation\ matrix\,'$

$\%'\ ob\ = \ [C;\ CA;\ CA\text{^}2;\ \ldots CA\text{^}(n-1)]. The\ system\ is\ completely\ state'$

$\%'\ observable\ if\ and\ only\ if\ o\ has\ a\ rank\ of\ n.$

$n = length(A);$

$for\ i = 1{:}\, n;$

$o(n + 1 - i, :) \ = \ C * A\text{^}(n - i);$

end

$if\ rank(ob) \sim = n$

$disp('System\ is\ not\ state\ observable')$

$else$

$disp('System\ is\ state\ observable')$

end

Example 6: Is the system given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y(t) = [1 \quad 1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$observable?$

Solution:

$>> \ A = [0\ 1\ 0; 0\ 0\ 1; -6\ \ -11\ \ -6];$

$>> \ C = [1\ 1\ 1];$

$>> \ ob(a, c)$

$System\ is\ state\ observable$

$ans\ =$

$$\begin{matrix} 1 & 1 & 1 \\ -6 & -10 & -5 \\ 30 & 49 & 20 \end{matrix}$$

That is, the system is $observable$ and the value of the matrix Q_o:

$$Q_o = \begin{bmatrix} 1 & 1 & 1 \\ -6 & -10 & -5 \\ 30 & 49 & 20 \end{bmatrix}$$

4. Decomposition of System State

From the discussion in the previous two sections, it is clear that the state variables (equivalently, the corresponding modes) of a linear system can generally be divided into the following four exclusive groups:

Case 1: Controllable and Observable

Case 2: Controllable but unobservable

Case 3: Uncontrollable but observable

case 4: Uncontrollable and unobservable.

Assuming that the system matrix A has different eigenvalues, the state equation can be simplified to the following form by appropriate transformation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u \qquad (4.1)$$

$$y = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The (transformed) system matrix A is put in "block diagonal" form, with each $A_i (i = 1,2,3,4)$ having a diagonal form. The suffix i of the state variable vector x means that the elements of this vector are the state variables corresponding to the i^{th} case defined above.

Example 7: Classify the state variables in a system defined by the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 2 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Solution: By inspection we can classify the state variables into the four groups as follows:

Case1: Controllable and observable, x_1, x_3 *and* x_6.

Case 2: Controllable and unobservable x_5.

Case 3: Uncontrollable and observable x_4.

Case 4: Uncontrollable and unobservable x_2.

We can represent the decompositions of the state variables into four groups by a diagram (see **Figure1**) showing the system divided into four subsystems each having state variables belong to one group only as indicated by the suffix i of S_i.

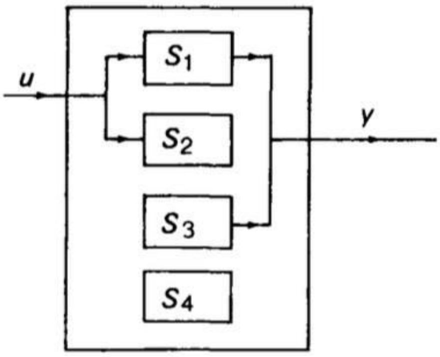

Figure 1: The system divided into four subsystems

This insight into the system structure explains the difference that may exist between the form of the system transfer function calculated from the system differential equations and that obtained by experimentation (that is, by obtaining the system frequency response.

We define the transfer function ($G(s)$ as $G(s) = \frac{Y(s)}{U(s)}$) as the ratio of the Laplace transform of the output $Y(s)$ to the input $U(s)$. The transfer function obtained from the differential equation (or equivalently from the system equation of state) includes all state variables (or modes) of the system. But the transfer function discovered through experimentation involves the part of the system that is affected by the input and affects the output. It can be seen from **Figure 1** that the transfer function of the subsystem Si is determined and only includes controllable and observable state variables (or modes).

In general, the transfer function $G(s)$ represents only the subsystem s_1 of the considered system, and indeed on adding to s_1 the subsystems S_2 , S_3 and S_4 has no effect on $G(s)$.

Example 8: Make sense of the above discussion by using the systems that are examined in example 1 and Decomposition of System State.

Solution: In example 1, the state equations were transformed into the diagonal form

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$y = [-1 \quad -4]z.$

There are two modes in the system, which correspond to the poles $\lambda = 1$ and $\lambda = 2$. Figure 2 can be used to represent the equations .

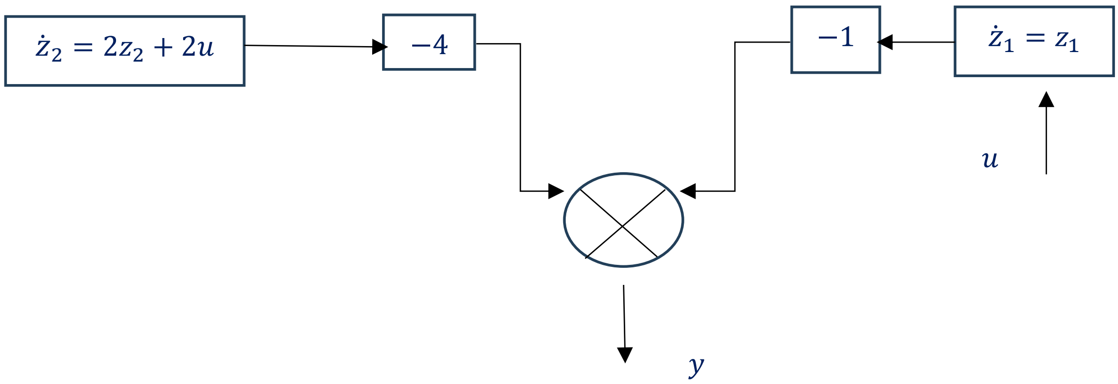

Figure 2: The representation of the equations in example 1

In Example 1 the transformed state equations are:

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} z_1 \\ z \end{bmatrix}$$

Figure 3 can be used to represent these equations.

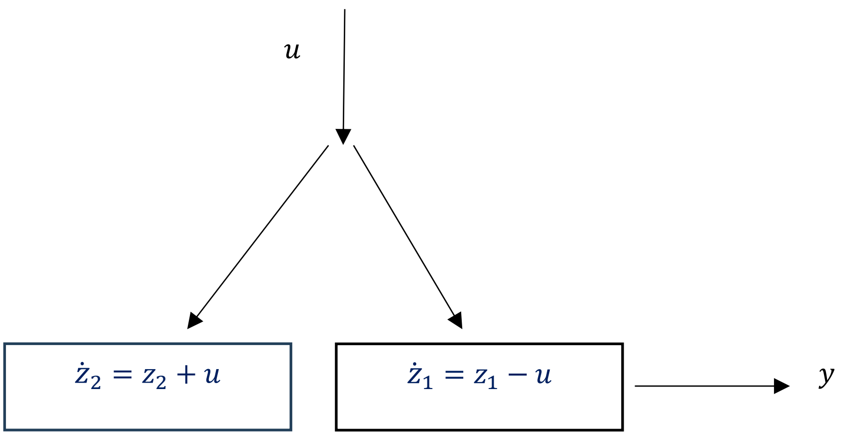

Figure 3: The representation of the equations

In this case

$$G(s) = [1 \quad 0]\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{s+1}$$

Once more, it is evident that the transfer function does not include the unobservable mode, which corresponds to the pole $\lambda = 1$.

It was mentioned in the example above that the unpredictable or the transfer function of a system lacks the unobservable mode. This fact warrants more investigation because it raises another requirement for a system to meet in order to be considered observable and or controllable [11]. It is believed that matrix A is of order n X n, with distinct eigenvalues .

5. Conclusion

This article discusses new methods of controllability and observability in control systems and the creation of algorithms using the MATLAB language in order to facilitate knowledge of controllability and observability in control systems, and the creation of applied methods as well. There are many well-known pieces of evidenceand methods that were not presented in this article. However, much of the proof is known, there is no doubt that there is much more undiscovered and that it is worth discussing, sharing, and presenting. Future research may address other things in depth.

6. References

[1] Farhan Hashosh, Ali, and Hadi Basirzadeh. "Routh stability criterion and Lyapunov-Routh method in control theory." International Journal of Nonlinear Analysis and Applications 15.5 (2024): 111-120.

[2] Hashoosh, Ali Farhan, and Hadi Basirzadeh. "Stability on Optimal control problems and Multi-Population Mathematical Models." Utilitas Mathematica 120 (2023): 500-505.

[3] R. E. KALMAN [1960a] A new approach to linear filtering and prediction problems, J. Basic Engr. (Trans. ASMS), 82D:35–45. YAN-MING FU; GUANG-REN DUAN; SHEN-MIN SONG, Design of

[4] BARTOSIEWICZ Z., PIOTROWSKA E., M. WYRWAS, Stability, Stabilization And Observers of Linear Control Systems on Time Scales, Proc. IEEE Conf. on Decision and Control, New Orleans, LA, December 2007, 2803–2808.

[5] Burghs, David N., and Alexander Graham. Control and optimal control theories with applications. Horwood Publishing, 2004.

[6] Unknown Input Observer for Linear Time-delay Systems. International Journal of Control, Automation, and Systems, vol. 2, No. 4, 2004, 530-535.

[7] Golnaraghi, Farid, and Benjamin C. Kuo. Automatic control systems. McGraw-Hill Education, 2017.

[8] SAADAT H., ComputationalAids in Control Systems Using MATLAB, Copyright-McGraw-Hill inc (1993).

[9] W.H. Fleming, R.W. Rishel, Deterministic and Stochastic Optimal Control, Springer-Verlag, New York, 1986.

[10] D'Azzo, J. J., Linear Control System Analysis, McGraw-Hill, 1988.

[11] D.Y. Karamzin, V.A. Oliveira, F.I. Pereira, G.N. Silva, On some extension of optimal control theory, Eur. J. Control 20 (2014) 284–291.