Special Methods Controllability and Observability in Optimal Control Systems

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Abstract:

This article explores a specific case study that examines controllability and observability in general and in specific so that it is easy for the reader to understand these two concepts, which are fundamental in optimal control theory. Algorithms have been written to determine the controllability and observability of optimal control systems using the MATLAB programming language, and new techniques have been developed to deal with them. In addition, a critical test was created in which the state variables of the system or, more precisely, their corresponding states were split together, illustrating this with more than one example. Divided into four groups in a linear manner, as the article explains. To understand the controllability and observability of some more complex systems, this article is a starting point for the future expansion of these two concepts through the development of new algorithms or other applied solution methods or the creation of new algorithms.

Keywords: Mathematical Modeling, Optimal Control, Controllability, Observability.

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1. Introduction:

Controllability and observability are two basic and important concepts in modern control theory. The stability of the control system and its types is also stable (Routh stability criterion and Lyapunov) [1] [2]. In 1960, Kalman defined these two ideas in order [2] to determine the degree to which a system can be observed and managed [3]. The following fundamental queries must be addressed for any control system, especially for multivariable systems:

a) Is it possible to find a control function $u(t)$ that will, in a finite amount of time, change the system's initial state ($x_t$) into the desired final state ($x_f$)?

b) Is it possible to assess the system's state by analyzing its performance over a limited time?

The terms controllability and observability refer to the two ideas at play. Accordingly, the system is controllable if the first question has a "yes" response. The system is also observable if the second question has a yes response. It is important to acknowledge the fundamental nature of these problems. For instance, it makes no sense to attempt controlling a system by feedback of a state variable that permits the system's poles to be positioned arbitrarily unless the system is controllable. Similarly, attempting to reconstruct unmeasurable state variables of the system using so-called observers is futile unless the system is observable. In reality, controllability and
observability are two dual concepts that are intimately associated with the cancellation of zeros and poles in the transfer function of the system [5].

2. Controllability

We say about a system that it is controllable if and only if it is possible through the control vector to bring the system from the initial state $x(t_0) = x_0$ any final state $x(t_f) = x_f$ within a specified time $t > 0$.

In the case of nonlinear systems, these equations take the following form:

$$
\dot{x} = A(t)x(t) + B(t)u(t)
$$

$$
y = C(t)x(t) + D(t)u(t)
$$

In the case of linear systems fixed with time, the equations take the following form:

$$
\dot{x} = Ax(t) + Bu(t)
$$

$$
y = Cx(t) + Du(t)
$$

Where

$A$ is the state matrix of order $n \times n$, $B$ is the control matrix of order $n \times m$, $C$ is the output matrix of order $1 \times n$, $D$ is Direct transfer matrix of order $1 \times m$, $n$ the number of state vector states, $m$ the number of control vector states [4]. We consider a system described by the state equations:

$$
\dot{x} = Ax(t) + Bu(t) \quad (2.1)
$$

$$
y = Cx(t)
$$

With the transformation:

$$
x(t) = pz(t) \quad (2.2)
$$

we can transform equation (2.1) into the form:

$$
\dot{z} = A_1 z(t) + B_1 u(t) \quad (2.3)
$$

$$
y = C_1 z(t)
$$

Where $A_1 = P^{-1}AP$, $B_1 = P^{-1}B$ and $C_1 = CP$. Assuming that $A$ has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ we can choose $P$ so that $A_1$ is a diagonal matrix, that is,

$$
A_1 = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}
$$

If $n = m = 2$, the first of the equations (2.3) has the form
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix} + \begin{bmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

Which is written as
\[
\dot{z}_1 = \lambda_1 z_1 + b_1^T u \\
\dot{z}_2 = \lambda_2 z_2 + b_2^T u
\]

Where \(b_1^T\) and \(b_2^T\) are the row vectors of the matrix \(B\).

The output equation is
\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\]

Which can be written as
\[
y_1 = c_{11} z_1 + c_{12} z_2 \\
y_2 = c_{21} z_1 + c_{22} z_2
\]

Or
\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
c_{11} \\
c_{21}
\end{bmatrix} z_1 + \begin{bmatrix}
c_{12} \\
c_{22}
\end{bmatrix} z_2
\]

So that
\[
y = c_1 z_1 + c_2 z_2
\]

where \(c_1\) and \(c_2\) are the column vectors of \(C\). So, in general, equation (2.3) can be written in the form:
\[
\dot{z}_i = \lambda_i z_i + b_i^T u(t) \quad (i = 1, 2, 3, \ldots, n)
\]

\[
y = \sum_{i=1}^{n} c_i z_i
\]

It is seen from equation (2.6) that if \(b_i^T\) the \(i^{th}\) row of \(B\) has all zero components, then
\[
\dot{z}_i = \lambda_i z_i + 0
\]

And the input \(u(t)\) has no influence on the \(i^{th}\) mode of the system. The mode is said to be uncontrollable, and a system having one or more such modes is uncontrollable [6][7].

**Example 1:** Check whether the system having the state-space representation
\[
\dot{x} = \begin{bmatrix}
-1 & 2 \\
-3 & 4
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
4 \\
6
\end{bmatrix} u \\
y = \begin{bmatrix}
1 & -2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Is controllable?

**Solution:** The characteristic equation is
\[
|\lambda I - A| = \lambda^2 - 3\lambda + 2 = 0
\]

\[
(\lambda - 1)(\lambda - 2) = 0
\]

\[
\Rightarrow \lambda = 1 \& \lambda = 2
\]
The corresponding eigenvectors are
\[ x_1 = [1 \ 1]^T \text{ and } x_2 = [2 \ 3]^T \]
so that the modal matrix is
\[ P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \]
Using the transformation \( x = Pz \), the state-equation becomes
\[
\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\
y = [-1 \ -4] z
\]
This equation shows that the first mode is uncontrollable and so the system is uncontrollable.

On the basis of the above result, we now derive an extremely useful criterion for determining whether a system is controllable. Although at this stage we consider only the necessity of this criterion, it is also a sufficient condition. To simplify the notation and the mathematical manipulations, we consider a SISO (single-input and single-output) system, so that in equation (2.1) \( B \) is a one-column matrix, that is a column vector \( b \) (say), and \( C \) is a row vector \( c_1^\top \). The result holds for the more general case when the system is multivariable.

Equations (2.1) and (2.3) are then written as
\[
\dot{x} = Ax(t) + bu(t) \quad (2.1.\ a) \\
y = Cx(t)
\]
And
\[
\dot{z} = A_1z(t) + b_1u(t) \quad (2.3.\ a) \\
y = c_1^\top z(t)
\]
We have chosen an indirect way of deriving the controllability criterion. It has the advantage of simplicity, but a penalty we paid or this is some loss in the logic behind the setting up of the criterion. We have established that the necessary condition for the system defined by equation (2.1.\ a) to be controllable is that the components of the vector \( b_1 = [\beta_1 \ \beta_2 \ \ldots \ \beta_n]^T \) in equation (2.3.\ a) are all non-zero.

In equation (2.3.\ a) the matrix \( A_1 = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) where the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are assumed distinct. Hence the matrix:
\[
\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{bmatrix}
\]
Has linearly independent columns, so that it is non-singular. It follows that the necessary condition to be controllable is that the (partitioned) matrix:

\[
Q_p = [b_1 \ A_1 b_1 \ A_1^2 b_1 \ \ldots \ A_1^{n-1} b_1] = \begin{bmatrix}
\beta_1 & \lambda_1 \beta_1 & \ldots & \lambda_1^{n-1} \beta_1 \\
\beta_2 & \lambda_2 \beta_2 & \ldots & \lambda_2^{n-1} \beta_2 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_n & \lambda_n \beta_n & \ldots & \lambda_n^{n-1} \beta_n
\end{bmatrix}
\]

(2.7)

The matrix \(Q_p\) is non-singular.

Since

\[A_1 = P^{-1}AP \text{ and } b_1 = P^{-1}b\]

We have

\[A_1 b_1 = P^{-1}AP P^{-1}b = P^{-1}Ab\]
\[A_1^2 b_1 = P^{-1}A^2P P^{-1}b = P^{-1}A^2b\]
\[
\vdots
\]
\[A_1^{n-1} b_1 = P^{-1}A^{n-1}P P^{-1}b = P^{-1}A^{n-1}b\]

So that

\[Q_p = P^{-1}[b \ Ab \ A^2b \ \ldots \ A^{n-1}b] = P^{-1}Q_c\]

Where

\[Q_c = [b \ Ab \ A^2b \ \ldots \ A^{n-1}b]\]

(2.8)

Since \(Q_p\) (for a controllable system) and \(P^{-1}\) are both non-singular, \(Q_c\) (for a controllable system) is also non-singular.

As \(Q_p\) is non-singular, its \(n\) columns are linearly independent. So that the rank of the matrix \(Q_c\) written as \(r(Q_c)\) is \(n\).

2.1. Controllability Test

To find out the controllability of a system, consider a system described by the state equations:

\[
\dot{x} = Ax(t) + Bu(t)
\]

\[y = Cx(t)\]

Step 1: write the matrix \(Q_c\) (is called the system controllability matrix)

\[Q_c = [B \ AB \ A^2B \ \ldots \ A^{n-1}B]\]

Step 2: find the determinant of \(Q_c\) if it is not equal to zero then the control system is controllable or if the determinant of \(Q_c\) equal to zero then the control system is uncontrollable.

If the matrix \(A\) in system order is higher than \(3 \times 3\), it is difficult to know whether the system is controllable or not in a previous way, so we assume that
\[ \Phi = [B \ AB \ A^2B \ \cdots \ A^{n-m}B] \]

When \( m \) is number of inputs

Step 1: write the matrix \( Q_c \) (is called the system controllability matrix)

\[ \Phi = [B \ AB \ A^2B \ \cdots \ A^{n-m}B] \]

Step 2: find rank of \( \Phi \) if it is equal to \( n \) then the control system is controllable or if rank of \( \Phi \) not equal to \( n \) then the control system is uncontrollable.

Or we can calculate the determinant \( \Phi \Phi^T \) if it is not equal to zero then the control system is controllable or if determinant of \( \Phi \Phi^T \) equal to zero then the control system is uncontrollable.

**Example 2:** Verify the controllability of control system which is the presented by state equation:

\[
\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & x_1 \\ 3 & 0 & 0 & 2 & x_2 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & -2 & 0 & 0 & x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u
\]

**Solution:**

Step 1: \( \Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & -4 \end{bmatrix} \)

Step 2: \( \text{rank}(\Phi) = 4 = n \)

Then the system is controllable.

**2.2. Program for finding controllability**

A program was developed in MATLAB to create controllability, and it was saved under the name “ctrb” and can be used when needed.

The program:

```
Function co = ctrb(A,B)
N = length(A);
co = ctrb(a,b);
if rank(co) \equiv n
    disp('no contrable')
else
    disp(' contrable')
end
```

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Example 3: Is the system given as follows controllable?

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = 
\begin{bmatrix}
-1 & 0 & 0 \\
-1 & -2 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u
\]

\[
y(t) = 
\begin{bmatrix}
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

Solution: Using the preceding method results in the following

\[ A = [-1 \ 0 \ 0; -1 \ -2 \ 0; 1 \ 0 \ 0]; \]

\[ B = [1; 0; 0]; \]

\[ co(A,B) \]

contable

\[ ans = \]

\[
\begin{bmatrix}
1 & -1 & 1 \\
0 & -1 & 3 \\
0 & 1 & -1
\end{bmatrix}
\]

That is, the system is controllable and the value of the matrix \( Q_c \):

\[
Q_c = 
\begin{bmatrix}
1 & -1 & 1 \\
0 & -1 & 3 \\
0 & 1 & -1
\end{bmatrix}
\]

3. Observability

A system is said to be observable if the initial vector \( x(t) \) can be found from the measurement of \( u(t) \) and \( y(t) \). The plant described by (2.1) is completely state observable if the inverse matrix exists [8].

By using the transform \( x = Pz(x) \) as in the (2.1) section, we end up with the system state equations in the form of equation (6.3), that is

\[
\dot{z} = A_1 z(t) + B_1 u(t)
\]

\[
y = C_1 z(t)
\]

If a row of the matrix \( C_1 \) is zero, the corresponding mode of the system will not appear in the output \( y \). In this case the system is unobservable, since we cannot determine the state variable corresponding to the row of zeros in \( C_1 \) from \( y \).
Example 4: Check whether the system having the state-space representation
\[
\dot{x} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u
\]
\[
y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]
Is observable?

Solution: The characteristic equation is
\[
|\lambda I - A| = \lambda^2 - 1 = 0
\]
\[
(\lambda - 1)(\lambda + 1) = 0
\]
\[
\Rightarrow \lambda = 1 & \lambda = -1
\]
The corresponding eigenvectors are
\[
x_1 = [1 \ 1]^T \ and \ x_2 = [2 \ 3]^T
\]
so that the modal matrix is
\[
P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \ and \ P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}
\]
Using the transformation \(x = Pz\), the state-equation becomes
\[
\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u
\]
\[
y = [1 \ 0] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]
Then the system is unobservable.

Example 5: illustrates the importance of the observability concept. In this case, we have an unstable system, whose instability is not observed in the output measurement. The dual controllability concept is of equal theoretical importance. An uncontrollable system has one or more modes that are not influenced by the input.
We now similarly derive a criterion for observability to that used to derive the controllability criterion.

Again, for simplicity, we consider a SISO system, but the result holds for the more general multivariable system. It can be seen that the necessary conditions for systems defined by equation (1.a) to be observable is that the components of the vector \(b_1 = [y_1 \ y_2 \ \cdots \ y_n]^T\) in equation (3.a) are all non-zero
for a controllable system we have the matrix
\[
Q_1 = \begin{bmatrix} c_1^T \\ c_1^T A_1 \\ \vdots \\ c_1^T A_1^{n-1} \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \lambda_1 & \cdots & y_n \lambda_1^{n-1} \\ y_2 & y_2 \lambda_2 & \cdots & y_n \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_n \lambda_1^{n-1} & \cdots & y_n \lambda_n^{n-1} \end{bmatrix}
\]
(3.1)
The matrix \(Q_1\) is non-singular.
Since  
\[ A_1 = P^{-1}AP \text{ and } C_1^T = c^T P \]

We have  
\[ c_1^T A_1 = c^T PP^{-1}AP = c^T AP \]
\[ A_2^2 b_1 = c^T PP^{-1}A^2 P = c^T A^2 P \]
\[ \vdots \]
\[ A_1^{n-1} b_1 = c^T PP^{-1}A^{n-1} P = c^T A^{n-1} P \]

So that  
\[ Q_1 = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} P = Q_o P \]

Where  
\[ Q_o = \begin{bmatrix} c^T \\ c^T A_1 \\ \vdots \\ c^T A_1^{n-1} \end{bmatrix} \]  \hspace{1cm} (3.2)

Since \( Q_o \) (for an observable system) and \( P \) are both non-singular, \( Q_o \) (for a observable system) is also non-singular.

3.1. The observability criterion

In equation (3.1) if the rank of matrix \( Q_o \) is \( n \), the system can be called observable system. If rank \( Q_o \) less than \( n \), the system is unobservable.

3.2. Observability test

A control system is said to be observable if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

To find out whether the control system is observable or not, we use a Kalman’s test:

Step 1: form the matrix \( Q_o = [C^T \ A^T \ C^T (A^T)^2 C^T \ \ldots \ \ (A^T)^{n-1} C^T] \)

Step 2: Take determinant of \( Q_o \) if it is not equal to zero then the control system is observable or if determinant of \( Q_c \) equal to zero then the control system is not observable [9].

Example 5: Verify the observability of control system which is the presented by state equation:

\[ \dot{x} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \]
\[ y = [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Solution: Given \( A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( c = [1 \ 1] \), \( n = 2 \)
Step 1: 
\[ Q_o = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \ldots \quad (A^T)^{n-1} C^T] = [C^T \quad A^T C^T] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \]

Step 2: 
\[ |Q_o| = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix} = -1 \]

the system is observable.

3.3. Program for finding observability

MATLAB program was designed to find observability and was saved under the name “obsv” It is used when needed, as shown below [10]:

```matlab
function ob = obsv(A, C)
    % The function ob = obsv(A, C) returns the transformation matrix
    % ob = [C; CA; CA^2; \ldots CA^{n-1}]. The system is completely state'
    % observable if and only if ob has a rank of n.
    n = length(A);
    for i = 1:n;
        o(n + 1 - i,:) = C * A^(n - i);
    end
    if rank(ob) == n
        disp('System is not state observable')
    else
        disp('System is state observable')
    end
end
```

**Example 6:** Is the system given as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 & -11 & -6
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} u
\]

\[ y(t) = \begin{bmatrix}
1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} \]

observable?

**Solution:**

```matlab
>> A = [0 1 0; 0 0 1; -6 -11 -6];
>> C = [1 1 1];
>> ob(a, c)
```

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System is state observable

\[
\text{ans } = \\
\begin{bmatrix}
1 & 1 & 1 \\
-6 & -10 & -5 \\
30 & 49 & 20
\end{bmatrix}
\]

That is, the system is observable and the value of the matrix \( Q_o \):

\[
Q_o = \begin{bmatrix}
1 & 1 & 1 \\
-6 & -10 & -5 \\
30 & 49 & 20
\end{bmatrix}
\]

4. Decomposition of System State

From the discussion in the previous two sections, it is clear that the state variables (equivalently, the corresponding modes) of a linear system can generally be divided into the following four exclusive groups:

Case 1: Controllable and Observable

Case 2: Controllable but unobservable

Case 3: Uncontrollable but observable

Case 4: Uncontrollable and unobservable.

Assuming that the system matrix \( A \) has different eigenvalues, the state equation can be simplified to the following form by appropriate transformation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & A_3 & 0 \\
0 & 0 & 0 & A_4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u 
\]

\[
y = \begin{bmatrix}
C_1 & 0 & C_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} 
\]

The (transformed) system matrix \( A \) is put in "block diagonal" form, with each \( A_i (i = 1,2,3,4) \) having a diagonal form. The suffix \( i \) of the state variable vector \( x \) means that the elements of this vector are the state variables corresponding to the \( i^{th} \) case defined above.

Example 7: Classify the state variables in a system defined by the following state equation:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
1 & -1 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
1 & 0 \\
0 & 2
\end{bmatrix} u 
\]
\[
\begin{bmatrix}
\gamma_1 \\
\gamma_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -1 & 2 & 0 & 1 \\
1 & 0 & 1 & -1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}
\]

**Solution:** By inspection we can classify the state variables into the four groups as follows:

Case 1: Controllable and observable, \(x_1, x_3\) and \(x_6\).
Case 2: Controllable and unobservable \(x_5\).
Case 3: Uncontrollable and observable \(x_4\).
Case 4: Uncontrollable and unobservable \(x_2\).

We can represent the decompositions of the state variables into four groups by a diagram (see Figure 1) showing the system divided into four subsystems each having state variables belong to one group only as indicated by the suffix \(i\) of \(S_i\).

![Figure 1: The system divided into four subsystems](image)

This insight into the system structure explains the difference that may exist between the form of the system transfer function calculated from the system differential equations and that obtained by experimentation (that is, by obtaining the system frequency response.

We define the transfer function \(G(s)\) as \(G(s) = \frac{Y(s)}{U(s)}\) as the ratio of the Laplace transform of the output \(Y(s)\) to the input \(U(s)\). The transfer function obtained from the differential equation (or equivalently from the system equation of state) includes all state variables (or modes) of the system. But the transfer function discovered through experimentation involves the part of the system that is affected by the input and affects the output. It can be seen from Figure 1 that the transfer function of the subsystem \(S_i\) is determined and only includes controllable and observable state variables (or modes).
In general, the transfer function $G(s)$ represents only the subsystem $s_1$ of the considered system, and indeed on adding to $s_1$ the subsystems $S_2, S_3$ and $S_4$ has no effect on $G(s)$.

**Example 8:** Make sense of the above discussion by using the systems that are examined in example 1 and Decomposition of System State.

**Solution:** In example 1, the state equations were transformed into the diagonal form

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -4 \end{bmatrix} z.$$  

There are two modes in the system, which correspond to the poles $\lambda = 1$ and $\lambda = 2$. Figure 2 can be used to represent the equations.

![Figure 2: The representation of the equations in example 1](image)

In Example 1 the transformed state equations are:

$$\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z \end{bmatrix}$$

![Figure 3: The representation of the equations](image)
In this case

\[
G(s) = \begin{bmatrix} 1 & 0 \\ \frac{s+1}{s-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = -\frac{1}{s+1}
\]

Once more, it is evident that the transfer function does not include the unobservable mode, which corresponds to the pole \( \lambda = 1 \).

It was mentioned in the example above that the unpredictable or the transfer function of a system lacks the unobservable mode. This fact warrants more investigation because it raises another requirement for a system to meet in order to be considered observable and or controllable [11]. It is believed that matrix A is of order n X n, with distinct eigenvalues.

5. Conclusion

This article discusses new methods of controllability and observability in control systems and the creation of algorithms using the MATLAB language in order to facilitate knowledge of controllability and observability in control systems, and the creation of applied methods as well. There are many well-known pieces of evidenceand methods that were not presented in this article. However, much of the proof is known, there is no doubt that there is much more undiscovered and that it is worth discussing, sharing, and presenting. Future research may address other things in depth.

6. References


